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Dilatation and Vortex Stretching Effects on Turbulence in One-Dimensional/Axisymmetric Flows

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A theoretical analysis is carried out to predict the amplification of turbulence in flows experiencing one-dimensional and axisymmetric dilatations. When three representative vortices are tracked, the variations of vortex radius and vorticity are calculated, and then the amplification of turbulence is obtained from them. When a vortex is tracked, conservation laws of mass and angular momentum of the vortex are used. For a one dimensionally compressed flow, the present analysis slightly underestimates the amplification of velocity fluctuations and turbulent kinetic energy, relative to that of rapid distortion theory in the solenoidal limit. For an axisymmetrically distorted flow, the amplification of velocity fluctuations and turbulent kinetic energy and the anisotropy depend not only on the density ratio, but also on the ratio of streamwise mean velocities, which represents streamwise vortex contraction/stretching. The amplification predicted by the present analysis is in excellent agreement with that by the rapid distortion theory. In both flows, the amplification of turbulence is dictated by the mean density ratio, whereas the anisotropy is primarily dictated by the flow boundary. However, streamwise vortex stretching/contraction alters the amplification slightly and the anisotropy significantly in the axisymmetric flow.

Nomenclature

k_d	=	ratio of mean densities, ρ_2/ρ_1
k_U	=	ratio of streamwise mean velocities, U_2/U_1
L	=	length or length scale of a vortex tube
M	=	Mach number
R	=	radius of a vortex tube
S	=	cross-sectional area of the flow boundary
U	=	mean velocity
α	=	amplification factor
γ	=	specific heat ratio
ρ	=	mean density
σ	=	rms of turbulent velocity
σ_{10}	=	rms of turbulent velocity at an upstream location for an isotropic flow
Ω	=	vorticity

Subscripts

1	=	conditions at an upstream location
2	=	conditions at a downstream location

Superscripts

C	=	cross-streamwise or transverse direction
S	=	streamwise direction
TKE	=	turbulent kinetic energy

I. Introduction

SINCE the 1930s, several theoretical attempts have been carried out to predict the amplification of turbulence in a flow experiencing compression in the contraction of a wind tunnel or converging–diverging nozzle. Using the conservation of energy and angular momentum of a rotating cylindrical fluid element, Prandtl[1] predicted the amplification of streamwise and spanwise turbulent velocities in a flow passing through the contraction of a wind tunnel. According to his analysis, streamwise and spanwise turbulent velocities were amplified by $1/k_U$ and $\sqrt{k_U}$, respectively, where k_U is the mean velocity amplification factor. In his analysis, the flow was incompressible. Taylor[2] also tried to calculate the turbulence amplification by predicting vorticity amplification using a theory based on the conservation of circulation for an inviscid fluid. For a flow with initially isotropic turbulence, the streamwise turbulent velocity was amplified by $1.5/k_U$ by the flow boundary contraction.

For a low/weak level of turbulence, Ribner and Tucker[3] and Ribner[4] predicted the amplification of turbulent velocities using linearized equations where dissipation was neglected. Their analysis is called linear interaction analysis (LIA), which was proposed by Lee et al.[5,6] In this analysis, the velocity fluctuation was treated as a wave, as in the work of Moore.[7] As will be discussed later in detail, the result of LIA is identical in the solenoidal limit to that of rapid distortion theory (RDT), which is a more popular theoretical approach.

The basic concept of RDT was proposed by Prandtl[1] and Taylor,[2] where turbulence distortion/deformation timescale is sufficiently smaller than the timescales of fluctuations. When the distortion rate of turbulence is much greater than the turbulence decay rate or energy cascade rate, all nonlinear terms, such as pressure-dilatation correlation and dilatational dissipation, can be neglected. Under these conditions, the governing equations are linearized by neglecting all nonlinear terms. For a very small turbulent Mach number, Durbin and Zeman[8] predicted the amplification of turbulent quantities through axial and axisymmetric homogeneous compressions by using RDT. For the two extreme cases of solenoidal and dilatational modes, the amplification of turbulent kinetic energy (TKE) was analytically predicted by RDT for flows with a finite turbulent Mach number.[9,10] The analyses of Cambon et al.[9] and Jacquin et al.[10] will be further discussed later. In addition to the two limiting cases, Gillet et al.[11] derived the amplification factors of TKE and dissipation for incompressible flows from the k–ε model, when the flow experiences rapid enough distortion. Simone et al.[12] used the RDT in a flow experiencing one-dimensional and isotropic compression and/or pure shear. The RDT was also used in two-dimensional supersonic turbulent boundary layer experiencing a rapid expansion by Jayaram et al.[13] Their analysis, however, was only valid immediately downstream of the last expansion fan.

Another theoretical approach is to track the behavior of representative vortices to predict the amplification of turbulence. In this analysis, the vortex tracking is conducted to evaluate the variations of the radius and vorticity of the representative vortices, which are later used to calculate the amplification of turbulence. Basically, this

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approach is similar to that of Prandtl.[1] The effect of the mean density variation was taken into account in the approach, whereas it was not in the analysis of Prandtl. When the conservation of angular momentum of a vortex was used as Prandtl[1] did in incompressible flows, the amplification of velocity fluctuations in the streamwise and normal directions was successfully predicted for a supersonic boundary layer experiencing compression and expansion.[14] The analysis in Ref. 14 for the streamwise vorticity clearly shows the effect of vortex stretching/contraction and/or dilatation on vorticity. The effect of dilatation on vorticity was much greater than that of streamwise vortex stretching or contraction. The term vortex contraction refers to negative stretching or shortening of a vortex tube in the present paper.

The purpose of the present study is to further the research of Kim et al.[14] and to provide physical insight into the mechanism of the amplification of turbulence and TKE in flows experiencing axial and axisymmetric dilatation. The theoretical result of Ribner and Tucker[3] is to be reinterpreted to compare with the present analysis.

II. Analysis

The present analysis tracks three representative vortices to evaluate the variations in the radius and vorticity of the three vortex filaments. Once these two quantities are obtained for each vortex, the amplification of turbulent velocities and TKE is calculated from them. In this sense, this analysis can be called vortex tracking analysis. The velocity fluctuation in one direction is caused by the vortices normal to the direction. For example, the streamwise velocity fluctuation is induced by cross-streamwise vortices. Thus, by tracking three representative vortices aligned with three corresponding directions of the coordinate system, the amplification of turbulent quantities can be obtained. Because the amplification quantities are calculated from the variation in vorticity of three vortices in this analysis, the interaction between vorticies are not taken into account as in RDT. Also the effects of viscosity are neglected in the analysis.

The two conservation equations of mass and angular momentum are used to calculate the variation of the radius and vorticity of a given vortex. The mass conservation law is applied first to evaluate the radius after the dilatational distortion. Then the vorticity variation due to the distortion is calculated by using the conservation law of the angular momentum of the vortex filament as in the work of Kim et al.[14] When compared to the RDT, this analysis is relatively simple and straightforward. However, this analysis helps to understand the roles of the mean flow parameters, for example, the mean density and vortex stretching, in turbulence and anisotropy.

A. Axial Compression/Expansion

1. *Amplification of Vorticity*

When a vortex in a flow experiences axial or one-dimensional dilatation, its dimensions in cross-stream directions remain the same as shown in Fig. 1. Thus, an equation of the conservation of mass for a streamwise vortex filament is written as

$$\rho_1 L_1^S L_1^C L_1^C = \rho_2 L_2^S L_2^C L_2^C \tag{1}$$

For the streamwise vortex filament, $L^C L^C$ is proportional to the square of the radius or the cross-sectional area of the vortex, that is,

$$\left(L_2^C / L_1^C\right)^2 = \left(R_2^S / R_1^S\right)^2 \tag{2}$$

which means that the product of the mean density and the length of the vortex is preserved. Because the dimensions in directions normal to the dilatation direction are kept the same, the radius of the vortex before the distortion equals that after the distortion, that is, $R_1^S = R_2^S$. From this and Eq. (1), one can show that

$$\rho_1 L_1^S = \rho_2 L_2^S \tag{3}$$

When the continuity equation of the mean flow $\rho_1 U_1^S S_1 = \rho_2 U_2^S S_2$ is considered, L_2^S / L_1^S is equal to U_2^S / U_1^S because the cross-sectional area of the flow boundary S is constant for the one-dimensional flow. As will be shown later, the relation $L_2^S / L_1^S = U_2^S / U_1^S$ holds also for the streamwise vortex in two-dimensional flows.

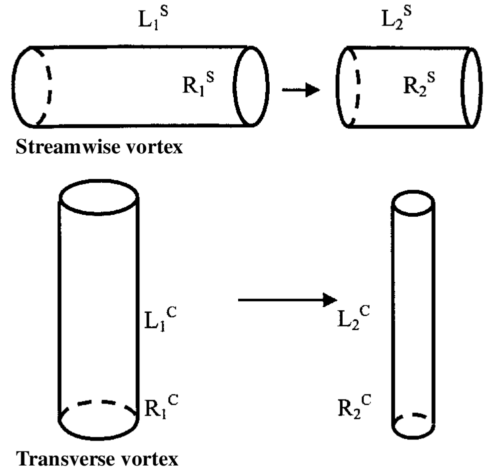

Fig. 1 Schematic of vortices experiencing axial or one-dimensional compression.

From the conservation of angular momentum of a vortex filament with a radius R, one can show that

$$\left(R_1^S\right)^2 \Omega_1^S = \left(R_2^S\right)^2 \Omega_2^S \tag{4}$$

From $R_1^S = R_2^S$ for the streamwise vortex in a one-dimensional flow, one can also show that streamwise vorticity does not vary for axial dilatation:

$$\Omega_2^S / \Omega_1^S = \left(R_1^S / R_2^S\right)^2 = 1 \tag{5}$$

These relations can be used for axially compressed and expanded flows.

For the cross-streamwise vortex, the equation of mass conservation is

$$\rho_1 \left(R_1^C\right)^2 L_1^C = \rho_2 \left(R_2^C\right)^2 L_2^C \tag{6}$$

The length of the transverse vortex filament remains unaltered for the axially compressed/expanded case, that is, $L_1^C = L_2^C$, and thus, the ratio of radii is

$$R_2^C / R_1^C = (\rho_1/\rho_2)^{\frac{1}{2}} = 1 / \sqrt{k_d} \tag{7}$$

where $k_d = \rho_2/\rho_1$. When Eqs. (5) and (7) are used, the amplification of cross-streamwise vorticity is

$$\Omega_2^C / \Omega_1^C = \left(R_1^C / R_2^C\right)^2 = k_d \tag{8}$$

2. *Amplification of Velocity Fluctuations*

The velocity fluctuation in one direction is generated by the other two vortices aligned in the other two directions, which are normal to the velocity direction, as stated earlier. For example, the streamwise velocity fluctuation is generated by both spanwise and normal or cross-streamwise vortices. Thus, streamwise and cross-streamwise rms velocity fluctuations σ^S and σ^C are, respectively, written as[14]

$$\sigma^S = \Omega^C R^C \tag{9}$$

$$\sigma^C = \sqrt{[(\Omega^S R^S)^2 + (\Omega^C R^C)^2]/2} \tag{10}$$

When Eq. (9) is used, the streamwise velocity fluctuation σ_2^S after expansion/compression is

$$\sigma_2^S = \Omega_2^C R_2^C \tag{11}$$

Substituting Eqs. (7) and (8) into the equation, one obtains

$$\sigma_2^S = \sqrt{k_d}\left(\Omega_1^C R_1^C\right)$$

From this equation, the amplification of rms streamwise velocity fluctuation $\alpha^S \equiv \sigma_2^S / \sigma_1^S$ is

$$\alpha^S = \sqrt{k_d} \tag{12}$$

where $\sigma_1^S = \Omega_1^C R_1^C$.

Similarly, the rms of cross-streamwise velocity fluctuation σ_2^C is

$$\sigma_2^C = \sqrt{\left[\left(\Omega_2^S R_2^S\right)^2 + \left(\Omega_2^C R_2^C\right)^2\right]/2}$$

Substituting Eqs. (5), (7), and (8) into the equation and using $R_1^S = R_2^S$, one can have

$$\sigma_2^C = \sqrt{\left[\left(\sigma_1^S\right)^2 + k_d\left(\sigma_1^C\right)^2\right]/2}$$

For initially isotropic turbulence, one can set $\sigma_{10} = \sigma_1^S = R_1^S\Omega_1^S = \sigma_1^C = R_1^C\Omega_1^C$, as Kim et al.[14] did, and thus, one obtains the amplification of rms of the cross-streamwise velocity fluctuation $\alpha^C \equiv \sigma_2^C/\sigma_{10}$ as

$$\alpha^C = \sqrt{(1 + k_d)/2} \tag{13}$$

The amplification of velocity fluctuation in the streamwise direction α^S is greater than that in the cross-streamwise direction α^C when the flow is compressed as shown in Eqs. (12) and (13) because k_d is greater than 1.

The amplification of TKE α^{TKE} is calculated by using Eqs. (12) and (13) as

$$\alpha^{\text{TKE}} = (1 + 2k_d)/3 \tag{14}$$

As the density ratio increases by compression, TKE is amplified in proportion to density ratio k_d monotonically. For expansion cases, the amplification factor approaches one-third as the density ratio k_d approaches zero. A more detailed discussion will follow.

B. Axisymmetric Dilatation

For this flow regime, the continuity equation (1) and the conservation equation of angular momentum (4) will be used again. As in the axial or one-dimensional dilatation case, two cross-streamwise components of length scales and amplification factors are the same. The tracking of each vortex is conducted in the same way as for the one-dimensional flow.

1. *Amplification of Vorticity*

For the streamwise vortex shown in Fig. 2, the radius of the vortex shrinks or expands approximately in proportion to the square root of the cross-sectional area of the flow boundary S:

$$R_2^S/R_1^S = (S_2/S_1)^{\frac{1}{2}} = (\rho_1 U_1/\rho_2 U_2)^{\frac{1}{2}}$$

where the continuity equation of the mean flow was used. By using density ratio k_d and streamwise velocity ratio $k_U = U_2/U_1$, one obtains

$$R_2^S/R_1^S = 1\big/\sqrt{k_d k_U} \tag{15}$$

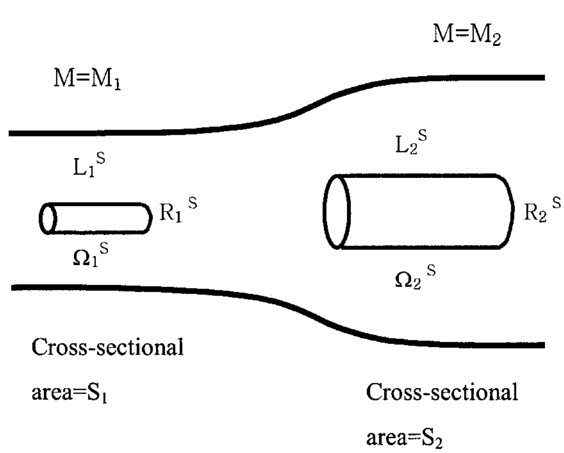

Fig. 2 Schematic of a streamwise vortex experiencing vortex stretching through expansion, in an axisymmetric flow.

Substituting this equation into Eq. (5), one can have

$$\Omega_2^S/\Omega_1^S = k_d k_U \tag{16}$$

This equation is identical to the relation $\Omega_2^S/\Omega_1^S = (\rho_2 U_2)/(\rho_1 U_1)$ obtained by Kim and Samimy[15] and Kim et al.[14] In a supersonic flow subjected to expansion, the streamwise vortex experiences vortex stretching and bulk dilation as well. Because the reduction of vorticity due to the mean density dilation is much greater than the gain of vorticity by stretching, the resultant streamwise vorticity decreases in an expanded supersonic flow, as discussed by Kim and Samimy.[15]

For a cross-streamwise vortex normal to the streamwise direction, the ratio of vortex tube lengths is approximately proportional to the square root of the cross-sectional area of the flow boundary:

$$L_2^C/L_1^C \approx \sqrt{S_2/S_1} = 1\big/\sqrt{k_d k_U} \tag{17}$$

Substituting this equation into the mass conservation equation of the transverse vortex filament, $\rho_1(R_1^C)^2 L_1^C = \rho_2(R_2^C)^2 L_2^C$, one obtains

$$R_2^C/R_1^C = (k_U/k_d)^{\frac{1}{4}} \tag{18}$$

When this equation and the conservation equation of angular momentum, $(R_1^C)^2\Omega_1^C = (R_2^C)^2\Omega_2^C$, are used, the ratio of vorticity in the cross-streamwise direction is given as

$$\Omega_2^C/\Omega_1^C = (k_d/k_U)^{\frac{1}{2}} \tag{19}$$

2. *Amplification of Velocity Fluctuations*

Now all quantities required for calculating the amplification of velocity fluctuations and TKE are obtained. For an initially isotropic turbulent flow, the amplification of turbulent velocity fluctuations is calculated from Eqs. (15), (16), (18), and (19). As in the preceding section, the amplification of TKE is obtained from the amplification factors of turbulent velocity fluctuations.

The streamwise velocity fluctuation is calculated by using Eqs. (11), (18), and (19):

$$\sigma_2^S = (k_d/k_U)^{\frac{1}{4}}\sigma_1^S \tag{20}$$

This results in the amplification of rms streamwise turbulent velocity $\alpha^S \equiv \sigma_2^S/\sigma_{10}$ as

$$\alpha^S = (k_d/k_U)^{\frac{1}{4}} \tag{21}$$

for an initially isotropic turbulent flow. The amplification factor in the cross-streamwise direction is obtained as

$$\sigma_2^C = \left\{\sqrt{\left[(k_d/k_U)^{\frac{1}{2}} + k_d k_U\right]/2}\right\}\sigma_{10} \tag{22}$$

by using Eqs. (15), (16), (18), and (19). This gives the amplification of turbulent velocity fluctuations in the cross-streamwise direction $\alpha^C \equiv \sigma_2^C/\sigma_{10}$:

$$\alpha^C = \sqrt{\left[(k_d/k_U)^{\frac{1}{2}} + k_d k_U\right]/2} \tag{23}$$

Finally, the amplification of TKE for axisymmetrically distorted flows is obtained using Eqs. (21) and (23) as

$$\alpha^{\text{TKE}} = \left(k_d k_U + 2\sqrt{k_d/k_U}\right)/3 \tag{24}$$

by the same way as in the one-dimensional flow.

From the mass conservation equation, one can show that $k_d k_U(S_2/S_1) = 1$. For a one-dimensional flow, the cross-sectional area remains the same, $S_2/S_1 = 1$, and thus, the mass conservation equation reduces to $k_d = 1/k_U$. This makes Eq. (24) identical to Eq. (14), which is the amplification of TKE in a one-dimensional flow.

III. Discussion

In this section, the theoretical results of the present analysis will be compared against those of RDT and/or LIA. In the present analysis, the interaction of vortices or fluctuating quantities is neglected as in the RDT, as stated earlier. Initially, the present analysis does not seem to need the requirement of rapid distortion of the flow to be applicable. When the interaction between fluctuating quantities is much smaller than dilatational effects, the present analysis can be applied. However, it is implicitly assumed in the present analysis that the energy spectrum of turbulence is simply shifted by compression/expansion. This requirement or assumption will be satisfied when the energy cascade rate is smaller than the distortion rate: inversely speaking, when the turbulence decay timescale is greater than the distortion timescale. Thus, the present analysis also needs the condition of rapid distortion of the flow although it does not strictly need the condition contrary to the RDT.

A. Axial or One-Dimensional Compression

An axial compression/expansion occurs in piston engines and shock tubes. For these cases, the dimensions of the flow boundary and vortices aligned in the transverse directions remain fixed. The amplification of velocity fluctuations, the square of α^S and/or α^C, is shown in Fig. 3 for axially compressed and expanded flows. When the present prediction curve is compared with that of RDT or LIA, it underestimates velocity fluctuations in both the streamwise and transverse directions. If air is compressed by a shock wave, the maximum density ratio is 6. In this case, the amplification predicted by the present analysis agrees well with that of Ribner and Tucker[3] or of the solenoidal limit of RDT. In the streamwise direction, the amount of underprediction is greater than that in the cross-streamwise direction. This results in greater anisotropy ($=\sigma^C/\sigma^S$) when compared to that of Ribner and Tucker, as shown in Fig. 4.

The amplification of TKE with increasing density ratio is shown in Fig. 5. Also shown in Fig. 5 are the solenoidal and pressure-released limits obtained from RDT.[9,10] The equation of the solenoidal limit was derived by RDT when the distortion Mach number $M_d = Dl/a$ (here D is the strain rate; l the integral length scale, and a the speed of sound), defined by Simone et al.,[12] was much smaller than unity ($M_d \ll 1$). Although the relation in the solenoidal limit for a compressed flow,

$$\alpha^{\mathrm{TKE}} = \frac{1}{2}\left(1 + k_d^2 \frac{\tan^{-1}\sqrt{k_d^2 - 1}}{\sqrt{k_d^2 - 1}}\right) \tag{25}$$

appears in Refs. 8–10 and 12, the equation can be easily obtained from the amplification factors of turbulent velocities by Ribner and Tucker.[3] Thus, the relation in the solenoidal limit by RDT is identical to that of Ribner and Tucker by LIA. Note from Eq. (25) that the TKE is monotonically amplified by compression and reduced by expansion. (In this case the equation needs to be modified slightly.) When the distortion Mach number is significantly greater than unity ($M_d \gg 1$), the amplification is limited by the pressure-released limit where pressure fluctuations are neglected.[9,10] This limit is also derived using the RDT and is written as

$$\alpha^{\mathrm{TKE}} = \left(2 + k_d^2\right)/3 \tag{26}$$

Gillet et al.[11] also obtained a theoretical equation predicting the amplification of TKE from the k–ε model equations when the flow distortion rate is much faster than the energy cascade rate, that is, when the conditions of rapid distortion are satisfied. The equation is shown as

$$\alpha^{\mathrm{TKE}} = k_d^{\frac{2}{3}} \tag{27}$$

These three curves defined by Eqs. (25–27) are shown in Fig. 5 for comparison. Equations (14), (25), and (27) are obtained under similar conditions: rapid distortion and negligible dissipation.

The amplification predicted by the present analysis is a little lower than that of the solenoidal limit of RDT, and the difference between them is less than 9.4% for a density ratio less than 6, which is the maximum value attainable by an adiabatic normal shock wave in air. The relation, which was obtained from the k–ε model by Gillet et al.,[11] significantly underestimates the amplification of TKE. Thus, the present analysis predicts the amplification of TKE better than that of Gillet et al.[11] In this flow, the amplification of TKE is independent of vortex stretching/compression, as can be inferred from Eqs. (14) and (27).

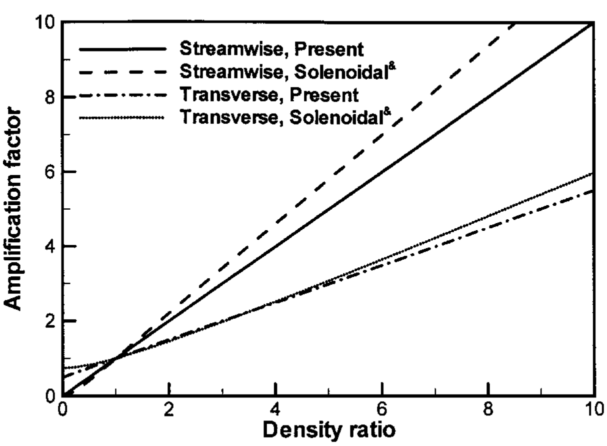

Fig. 3 Amplification factors of velocity fluctuations, square of α^S and/or α^C, with an increasing density ratio k_d in one-dimensional flow: compression for $k_d > 1$ and expansion for $k_d < 1$; curves for solenoidal limit (&) taken from Ribner and Tucker.[3]

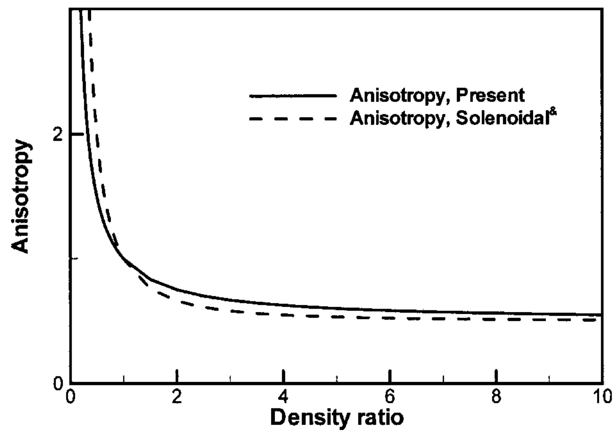

Fig. 4 Variation of anisotropy (σ^C/σ^S) with an increasing density ratio k_d in one-dimensional flow: compression for $k_d > 1$ and expansion for $k_d < 1$; curve for solenoidal limit (&) taken from Ribner and Tucker.[3]

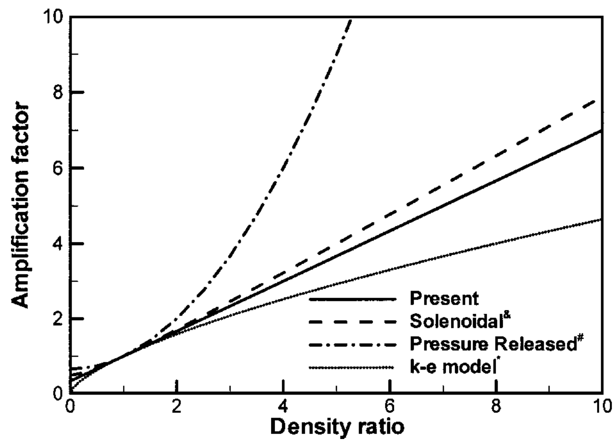

Fig. 5 Amplification factors of TKE with an increasing density ratio k_d in one-dimensional flow: compression for $k_d > 1$ and expansion for $k_d < 1$. Related references for the solenoidal limit (&), pressure limit (#), and k–ε model (*) are Ribner and Tucker,[3] Cambon et al.,[9] and Gillet et al.,[11] respectively.

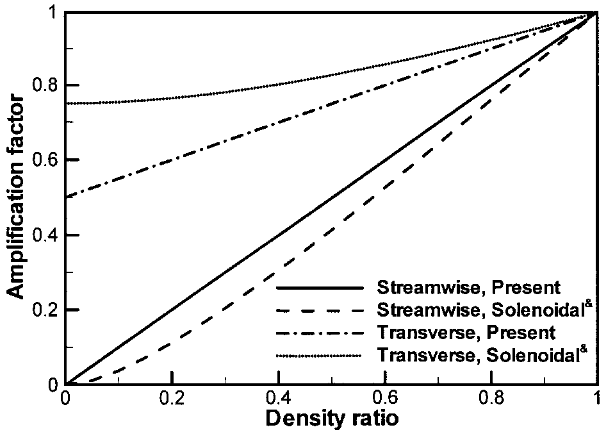

Fig. 6 Amplification factors of velocity fluctuations, square of α^S and/or α^C, for a one dimensionally expanded flow; curves for the solenoidal limit (&) taken from Ribner and Tucker.[3]

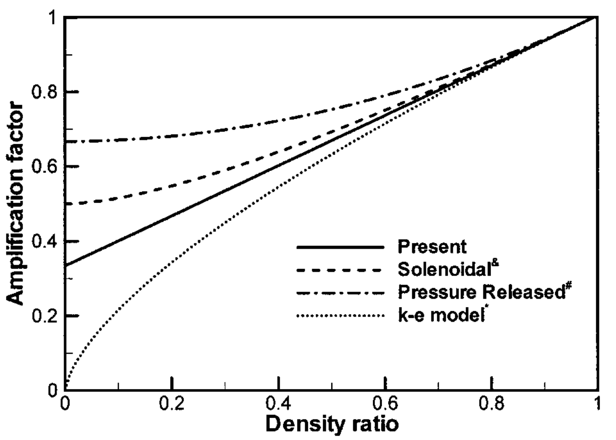

Fig. 7 Amplification of TKE α^{TKE} in a one dimensionally expanded flow. Related references for solenoidal limit (&), pressure released limit (#), and k-ε model (*) are Ribner and Tucker,[3] Cambon et al.,[9] and Gillet et al.,[11] respectively.

B. Axial Expansion

When the flow is axially expanded, the present analysis underestimates the cross-streamwise velocity fluctuations and TKE and overestimates the streamwise velocity fluctuation relative to the amplification predicted by LIA[3] or RDT,[9,10] as shown in Figs. 6 and 7. Contrary to the compression case, the discrepancy of the amplification factors, due to different approaches, is greater in the transverse direction than in the streamwise direction.

As will be shown later, the absolute values of velocity fluctuations and TKE approach zero with a vanishing density ratio in the axisymmetric flow regime. In the one-dimensional flow, the cross-streamwise velocity fluctuations and TKE, however, do not approach zero with a vanishing density ratio [see Eqs. (13) and (14)]. This is because of the invariant streamwise vorticity with axial distortion as shown in Eq. (5). The velocity fluctuation induced by the streamwise vortex is not amplified or reduced by an axial dilatation, and thus, this results in a finite fluctuation at a vanishingly small density ratio as shown in Fig. 7. This consideration is just from mathematical curiosity since the conservation equations used in the analysis probably may not hold at such a low density.

C. Axisymmetric Compression

The amplification factors of turbulent velocity fluctuations in streamwise and transverse directions are shown in Fig. 8 for an axisymmetrically compressed flow; the corresponding equations are Eqs. (21) and (23) for streamwise and cross-streamwise components, respectively. In Fig. 8, NS indicates that the amplification factor is obtained without taking account of vortex stretching or contraction, and STM and TRS represent streamwise and transverse directions, respectively. The upper and lower groups are for transverse and streamwise components, respectively. To obtain density

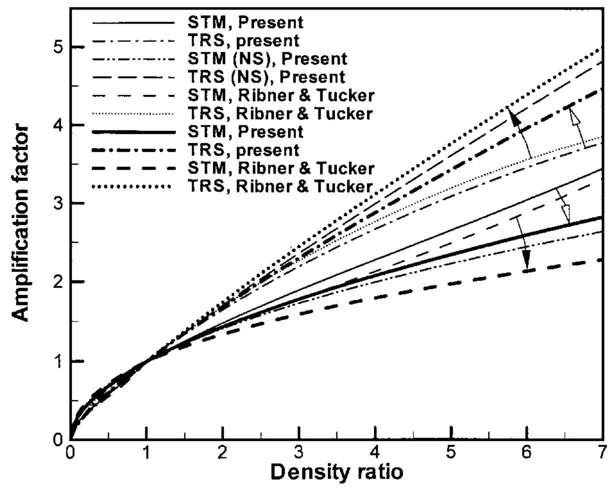

Fig. 8 Amplification factors of velocity fluctuations, square of α^S and/or α^C for an axisymmetric flow: compression for k_d >1 and expansion for k_d < 1; for thin curves, the upstream Mach numbers are 1 and 3 for expansion and compression cases, respectively, for the thick curves, corresponding Mach numbers are 3 and 5.

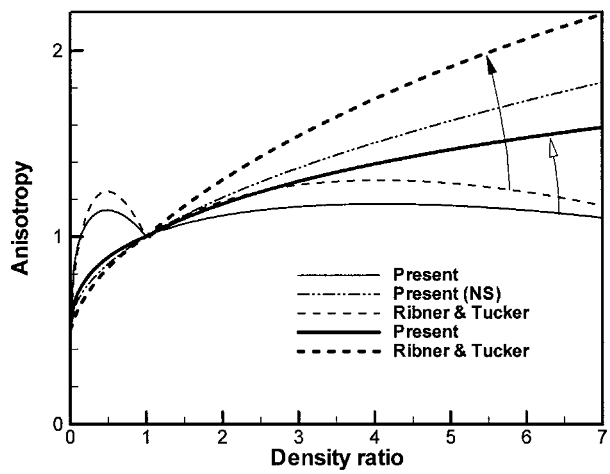

Fig. 9 Variations of anisotropy (σ^C/σ^S) with an increasing density ratio for an axisymmetric flow: compression for k_d >1 and expansion for k_d < 1; for thin curves, the upstream Mach numbers are 1 and 3 for expansion and compression cases, respectively, for the thick curves, corresponding Mach numbers are 3 and 5.

ratios, it was assumed that the flow was compressed by axisymmetric compression waves due to the smoothly contracted flow boundary. The LIA results of Ribner and Tucker[3] are also shown in Fig. 8. In both the present and the Ribner and Tucker analyses, the amplification of turbulent fluctuations and TKE is dependent not only on the density ratio but also on the streamwise velocity ratio, which is interpreted as streamwise vortex contraction or stretching. Recall that the amplification depended only on the density ratio in the one-dimensional flow.

A greater upstream Mach number results in a lower amount of streamwise contraction than a smaller upstream Mach number does. For this reason, each curve of the amplification factors of velocity fluctuations and anisotropy approach the corresponding curve as the upstream Mach number increases, as shown in Figs. 8 and 9. In Figs. 8 and 9, the arrows indicate the direction of the change of prediction curves with an increasing upstream Mach number: Smaller and larger Mach numbers are 3 and 5, respectively. When the effect of contraction on the streamwise vortex length is taken into account in the present analysis, the streamwise component of turbulent velocity fluctuations is greater than that predicted without considering the effect of contraction. On the other hand, the amplification of cross-streamwise velocity fluctuations becomes smaller due to the effect of contraction. This results in a decrease in anisotropy, defined as σ^C/σ^S, as shown in Fig. 9. However, the effects of streamwise vortex contraction on turbulent velocity fluctuations decrease with an increasing upstream Mach number as observed in Fig. 8. When the flow is compressed ideally by compression waves, the amplification of turbulent velocities depends on the upstream Mach number as well. The effect of the streamwise vortex contraction on velocity fluctuations decreases with an increasing upstream Mach number, as indicated by open arrows in Fig. 8. For a given density ratio, a different upstream Mach number results in a different contraction ratio in streamwise vortex. This makes the amplification of velocity fluctuations and anisotropy depend on the upstream Mach number. Both amplification curves approach the corresponding curves obtained by neglecting streamwise vortex contraction/stretching. The present analysis depends on the streamwise vortex contraction less than LIA or RDT, as shown in Figs. 8 and 9.

The streamwise component of velocity fluctuations predicted by the present analysis is greater than that by LIA of Ribner and Tucker,[3] whereas the transverse component is smaller than the corresponding component predicted by LIA. This results in a smaller anisotropy of the present analysis when compared to the Ribner and Tucker result as shown in Fig. 9. For a higher upstream Mach number of 5, the turbulence after the compression is less isotropic than for a smaller upstream Mach number of 3. With a decreasing upstream Mach number, the streamwise velocity ratio k_U decreases, and thus, the compression is more isotropic. For this region, the flow after the axisymmetric compression, with a lower upstream Mach number, becomes more isotropic than for a higher upstream Mach number.

The amplification of TKE is shown in Fig. 10 for a flow compressed ($k_d > 1$) and/or expanded ($k_d < 1$) axisymmetrically. As in the one-dimensional case, TKE increases monotonically with increasing density ratio. However, the amplification is smaller than that for the one-dimensional case for a given density ratio. Also shown is the prediction curve of Ribner and Tucker[3] for an axisymmetrically compressed flow. From their amplification factors of streamwise and transverse turbulent velocity fluctuations, the amplification of TKE is obtained as

$$\alpha^{\mathrm{TKE}} = \frac{1}{2}\left[k_d k_U + \left(1/k_U^2\sqrt{1-\varepsilon}\right)\tanh^{-1}\sqrt{1-\varepsilon}\right] \tag{28}$$

where

$$\varepsilon = 1/k_d k_U^3$$

Equation (28) is not shown in Ref. 3. By using their results of velocity fluctuations, the authors represented the amplification of TKE in terms of k_d and k_U, to compare with the present results.

Equations (24) and (28) clearly show that the amplification of TKE depends on the density and streamwise velocity ratios as mentioned earlier. The difference between the present and the Ribner and Tucker[3] prediction curves is very small, even negligible in an engineering sense. For a small upstream Mach number less than 3,

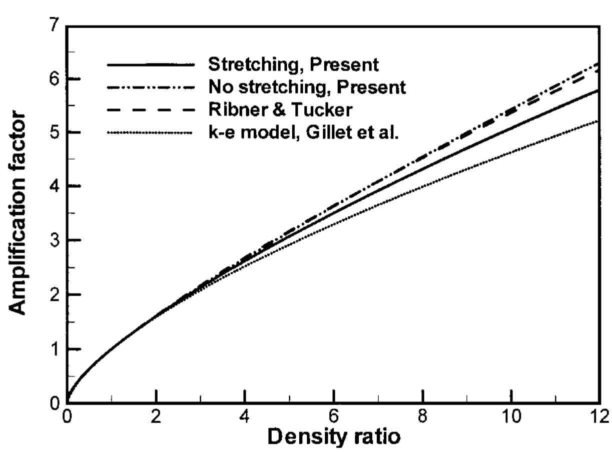

Fig. 10 Amplification factors of TKE α^{TKE} in an axisymmetric flow: compression for $k_d > 1$ and expansion for $k_d < 1$; upstream Mach numbers are 1 and 5 for expansion and compression cases, respectively.

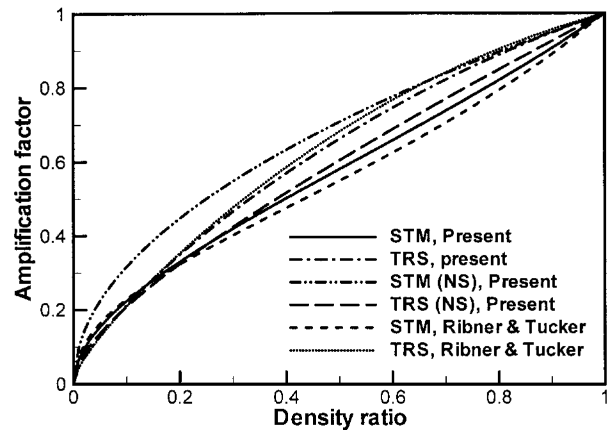

Fig. 11 Amplifications of velocity fluctuations in an axisymmetrically expanded flow with an upstream Mach number of 1.0.

it is hard to tell the difference. As the upstream Mach number increases, the prediction curve of Ribner and Tucker approaches that for no-contraction case of the present analysis.

As shown in Fig. 9, anisotropy is changed remarkably with streamwise vortex contraction. On the other hand, the effect of the streamwise vortex contraction on the amplification of TKE is very small even at a small upstream Mach number, as can be seen in Fig. 10. This negligible effect of the streamwise vortex contraction is associated with the fact that the increase in the streamwise velocity fluctuation, due to vortex contraction, offsets the decrease in the cross-streamwise velocity fluctuations, as shown in Fig. 8. From the negligible effects of streamwise vortex compression on TKE, it can be said that the amplification of TKE is primarily determined by the bulk compression or the mean density ratio. On the other hand, the anisotropy is strongly dependent on the flow boundary and streamwise vortex contraction/stretching.

D. Axisymmetric Expansion

When the flow experiences expansion, the equation of Ribner and Tucker[3] was modified to avoid a negative value in square root. Thus, the amplification equations, obtained from LIA or RDT, are different from those for the compression case, as in the axial or one-dimensional flow. However, a single form of an amplification relation was obtained for both compressed and expanded cases in the present analysis. As in the axisymmetrically compressed case, the streamwise vortex stretching altered turbulent velocity components and, thus, the anisotropy of turbulence as shown in Figs. 9 and 11, where the curves are at an upstream Mach number of 1. As the upstream Mach number increases, the amplification curves of the present and the Ribner and Tucker analyses approach the corresponding streamwise and transverse curves for the no-stretching cases of the present analysis. Also, the anisotropy approaches that for no-streamwise-stretching case.

However, the effects of the streamwise vortex stretching on the amplification of TKE is negligible, as shown in Fig. 12. This tendency is similar to the case of axisymmetrically compressed case. The amplification factor of TKE predicted by the present analysis excellently compares with that by LIA or RDT.

E. Application Limit

It is generally agreed that the compressible turbulence is composed of three modes, that is, vortical, acoustic, and entropic modes. In this analysis, it is assumed that the turbulence is quasi isentropic to exclude the contribution of entropic mode to the amplification of TKE, as was done by Durbin and Zeman,[8] Jacquin et al.,[10] and Cambon et al.[9] It is also assumed that the contribution of the acoustic mode to turbulence is negligible when compared to that of the vortical mode. Under these two conditions, the amplification of turbulence can be predicted by taking into account the vortical mode only.

The vortical, acoustic, and entropic modes are independent of each other in the linear inviscid limit and in the absence of mean

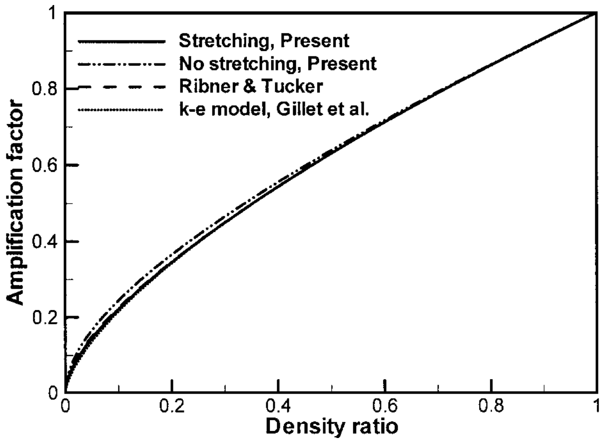

Fig. 12 Amplification factors of TKE in an axisymmetrically expanded flow with an upstream Mach number of 1.5.

gradients or body forces.[16,17] In such a case, the present analysis can be used to evaluate the contribution of the vortical mode to the turbulence field.

The present analysis is valid for homogeneously compressed flows, such as in a piston engine, and a flow compressed by a shock, such as in a shock tube. The results of Simone et al.[12] suggest that the contribution of nonlinear terms to the amplification of TKE is not significant in a wide range. In other words, the amplification of turbulence can be predicted by a linear analysis with a good accuracy. Thus, the present analysis is valid as far as the vortical mode of turbulence is dominant over the other modes.

The present analysis can be used to estimate turbulence variations in the cylinder of a piston engine, where one-dimensional dilatation takes place. Also, the reduction of relative turbulence intensity in a wind-tunnel contraction can be estimated by using the present analysis. If the present analysis is incorporated in a turbulence model, the anisotropic aspect of turbulence can be taken into account relatively easily.

IV. Conclusions

A theoretical analysis was carried out to predict the amplification factors of turbulent quantities in flows experiencing one-dimensional and axisymmetric dilatation. The idea of the present analysis is that, if one can predict the variations of vorticity and radius of three representative vortices, one can also predict the amplification of turbulent velocity fluctuations and turbulent kinetic energy by tracking each vortex for an initially isotropic turbulent flow. In tracking a vortex, the conservation laws of mass and angular momentum of the vortex were used to obtain the ratios of radii and vorticity through the dilatation. When the vortical mode is dominant over other modes of turbulence, the amplification of turbulence can be predicted by taking into account the vortical mode only.

For the one-dimensional flow, the relations predicting the amplification of turbulent velocities and TKE were obtained from the conservation equations of mass and angular momentum of a vortex, which experiences either expansion or compression dilatation. When the distortion Mach number was much smaller than unity (in the solenoidal limit), the present analysis slightly underestimated the amplification of turbulent velocities and TKE. For a density ratio less than 6, the agreement between the present analysis and RDT or LIA was quite good. In both analyses, the amplification factors and anisotropy depended only on the density ratio.

For an axisymmetrically distorted flow, the amplification factors of turbulent velocities and TKE and the anisotropy depended not only on the density ratio but also on the ratio of streamwise mean velocities, which represents streamwise vortex contraction/stretching. Although the effects of the streamwise vortex contraction on the TKE were negligible, those on the anisotropy of the turbulence were not negligible. The present analysis slightly overestimated the streamwise velocity fluctuation and underestimated the transverse velocity fluctuations, and thus underestimated the anisotropy when compared to the Ribner and Tucker[3] analytic results. However, the agreement in the amplification of TKE was excellent.

In both one-dimensional and axisymmetric flows, the bulk dilatation most probably governed the amplification of turbulent fluctuations for a given flow regime. As far as the anisotropy is concerned, the flow boundary appeared to play an important role. In the one-dimensional flow, the anisotropy was independent of streamwise vortex stretching or contraction. However, the anisotropy in the axisymmetric flow showed its strong dependence on the streamwise vortex stretching/contraction as mentioned earlier.

In obtaining the relations predicting the amplifications of turbulence, it was assumed that the interaction between fluctuating quantities and viscous dissipation are negligible as in the RDT. Thus, the results presented hold for an initially isotropic flow with rapid enough distortion rate and with negligible dissipation.

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